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MATHEMATICAL SIMULATION OF SEMITRANSPARENT LAYERED THIN-WALLED WAVEGUIDES WITH THE USE OF AVERAGED TWO-SIDED BOUNDARY CONDITIONS

V. T. Erofeenko and A. I. Glushtsov

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The boundary-value problem on the penetration of electromagnetic waves from a thin-walled cylindrical waveguide into the environment was solved with the use of averaged boundary conditions. The multilayer wall of the waveguide was considered as an ideally thin cylindrical surface, the complex material structure of which was described with the use of special two-sided boundary conditions. The dispersion equation determining the constants of propagation of partial waves in the waveguide was solved by the method of numerical minimization of the function of two variables. Results of calculation of the attenuation of the electromagnetic field penetrating from the waveguide into the environment as compared to the field of the corresponding mode inside the waveguide are presented.

Keywords: *layered waveguide, Maxwell equations, spectral problem, electromagnetic mode, attenuation coefficient.*

Introduction. An important problem of electrodynamics is the calculation of waveguide structures [1, 2], and, in the technical electrodynamics, of importance is the calculation of cable communication lines. In the last few decades the problems on the electromagnetic compatibility of technical means, in particular on the protection of communication lines from the action of external-noise carrying fields [3, 4] and the protection of the information carried by these lines from the nonsanctioned access [5] have come into importance. The waveguide communication lines have a layered material structure; therefore, the study of them with consideration for exact conditions at each boundary surface between the media is a difficult mathematical and computational problem [2, 6]. In this connection, interest has been shown in the use of approximate two-sided boundary conditions for calculation of waveguides [7] and investigations on the averaged boundary conditions [8] and the limits of their applicability [9] have appeared.

In recent years the protection of the information carried by communication lines has become a pressing problem [10]. In the present work, we estimated the penetration of the electromagnetic fields of different modes, propagating in a thin-walled waveguide, through its layered shell into the environment. The calculations were carried out with the use of approximate averaged two-sided boundary conditions. We determined the constants of propagation of the E_{nm} and H_{nm} modes in the waveguide as well as the coefficients of attenuation of their fields as a result of the penetration of them through the layered waveguide walls.

Formulation of the Problem. A vector boundary-value problem with classical boundary conditions was formulated for a waveguide. This problem was transformed into a boundary-value problem with averaged boundary conditions at the waveguide walls. Then it was formulated as a scalar boundary-value problem for the Helmholtz equation.

Spectral problem with exact boundary conditions. Let us consider a thin-walled circular cylindrical waveguide representing a tube with a wall of thickness Δ, an inner radius R_1, and an outer radius R_2 ($\Delta << R_1$), directed along the Oz axis. The wall of the waveguide D_0 consists of N cylindrical layers of thickness Δ_s $\left(\Delta = \sum_{s=1}^{N} \Delta_s\right)$ made from the material with electromagnetic parameters $\varepsilon_s^{\text{lay}}$, μ_s^{lay}, and γ_s^{lay}, where the numbering of layers $s = 1, 2, ..., N$ corresponds to the positive direction of the radial coordinate ρ of the cylindrical coordinate system ρ, φ, z. The medium inside the waveguide $D_1 = \{0 \le \rho < R_1\}$ is characterized by the parameters ε_1, μ_1, and γ_1, and the parameters of the medium out-

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side the waveguide $D_2 = \{\rho > R_2\}$ are ε_2, μ_2, and γ_2 ($|k_j| << |k_s^{\text{lay}}|$). An electromagnetic field with complex amplitudes $\mathbf{E}_1$ and $\mathbf{H}_1$ is formed in the region D_1; it penetrates through the waveguide wall (the cylindrical layer D_0), with the result that an electromagnetic field with complex amplitudes $\mathbf{E}_2$ and $\mathbf{H}_2$ is formed in the region D_2. In this case, classical boundary conditions of continuity of the tangential components of the electric and magnetic fields at the cylindrical boundary surfaces of the layers are fulfilled. It is proposed to solve the Maxwell equations

$$\operatorname{rot} \mathbf{E}_j = i\omega\mu_j \mathbf{H}_j\,, \quad \operatorname{rot} \mathbf{H}_j = -\,i\omega\varepsilon_j' \mathbf{E}_j\,, \quad \varepsilon_j' = \varepsilon_j + i\,\frac{\gamma_j}{\omega}\,, \tag{1}$$

in the regions D_j (j = 1, 2) in the form

$$\mathbf{E}_j = \mathbf{E}_j(\rho)\exp i\,(m\varphi + \beta z)\,, \quad \mathbf{H}_j = \mathbf{H}_j(\rho)\exp i\,(m\varphi + \beta z)\,, \quad \beta = \text{const}\,, \quad m = 0, \pm 1, \ldots\,, \tag{2}$$

(the solutions obtained are called the electromagnetic modes of the waveguide) and to calculate the spectral parameter $\beta = \beta_{mn}$ (n = 1, 2, ...), where β_{mn} are propagation constants representing complex quantities in the general case and ω is the circular frequency of the field.

Broadly speaking, the problem formulated can be solved with the use of the classical boundary conditions at the interfaces between the layers and with corresponding transmission matrices. In what follows we will use the method of averaged boundary conditions that makes it possible to simplify the procedure of calculating the propagation constants of the waveguide.

Spectral problem with averaged boundary conditions. Let us reformulate the initial spectral problem with the use of approximate boundary conditions called the averaged boundary conditions. To do this, we will change the cylindrical layer D_0 having a fairly small thickness for a middle ideal cylindrical surface Γ of radius $R = (R_1 + R_2)/2$ and will extend the region D_j to the surface Γ: $D_1 = \left\{0 \le \rho < R_1 + \frac{\Delta}{2}\right\}$, $D_2 = \left\{\rho > R_2 - \frac{\Delta}{2}\right\}$. At the surface Γ, we set the following averaged two-sided boundary conditions accounting for the material structure of the layer D_0 and coupling the fields $\mathbf{E}_j$ and $\mathbf{H}_j$ formed on each side of the surface Γ [8, p. 124]:

$$\Big[\mathbf{H}_2 - \mathbf{H}_1, \mathbf{n}\Big] = \Big[\mathbf{n}, [q_1\mathbf{E}_1 + q_2\mathbf{E}_2, \mathbf{n}]\Big]\,, \quad \Big[\mathbf{E}_2 - \mathbf{E}_1, \mathbf{n}\Big] = -\Big[\mathbf{n}, [p_1\mathbf{H}_1 + p_2\mathbf{H}_2, \mathbf{n}]\Big]\,, \tag{3}$$

where $\mathbf{n} = \mathbf{e}_\rho$ is the outer unit normal to the cylindrical surface Γ; e_ρ, e_φ, and e_z are the unit vectors of the cylindrical coordinate system; p_j and q_j are complex constants characterizing the material structure of the cylindrical shell D_0. The boundary conditions (3) were obtained in the locally-plane-wave approximation.

Thus, it is proposed to find solutions of Eqs. (1) in the form of (2) in the regions D_j (and the values of the spectral parameter β), satisfying the boundary conditions of conjugation (3) at the surface Γ, and the fields $\mathbf{E}_2$ and $\mathbf{H}_2$ in the region D_2, satisfying the condition of radiation at infinity.

Let us present the vector problem formulated as the boundary-value problem for the scalar potentials u and v. In our case, the solutions of the form $\mathbf{E}_j = \dot{\mathbf{E}}_j(\rho, \varphi)\exp(i\beta z)$, $\mathbf{H}_j = \dot{\mathbf{H}}_j(\rho, \varphi)\exp(i\beta z)$, β = const are determined. It is known that, in this case, all the cylindrical components of the field are expressed in terms of their z protections $u_j = \dot{E}_{jz}(\rho, \varphi)$ and $v_j = \dot{H}_{jz}(\rho, \varphi)$ [11]:

$$\dot{E}_{j\rho} = c^{(j)}\frac{\partial u_j}{\partial\rho} + \frac{b^{(j)}}{\rho}\frac{\partial v_j}{\partial\varphi}\,, \quad \dot{E}_{j\varphi} = \frac{c^{(j)}}{\rho}\frac{\partial u_j}{\partial\varphi} - b^{(j)}\frac{\partial v_j}{\partial\rho}\,, \quad \dot{H}_{j\rho} = c^{(j)}\frac{\partial v_j}{\partial\rho} - \frac{a^{(j)}}{\rho}\frac{\partial u_j}{\partial\varphi}\,, \quad \dot{H}_{j\varphi} = \frac{c^{(j)}}{\rho}\frac{\partial v_j}{\partial\varphi} + a^{(j)}\frac{\partial u_j}{\partial\rho}\,, \tag{4}$$

$$\dot{\mathbf{E}}_j = \dot{E}_{j\rho}\mathbf{e}_\rho + \dot{E}_{j\varphi}\mathbf{e}_\varphi + \dot{E}_{jz}\mathbf{e}_z\,, \quad \dot{\mathbf{H}}_j = \dot{H}_{j\rho}\mathbf{e}_\rho + \dot{H}_{j\varphi}\mathbf{e}_\varphi + \dot{H}_{jz}\mathbf{e}_z\,,$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\varphi$, and $\mathbf{e}_z$ are unit vectors of the cylindrical coordinate system;

$$a^{(j)} = \frac{i\omega\varepsilon_j'}{a_j^2}, \quad b^{(j)} = \frac{i\omega\mu_j}{a_j^2}, \quad c^{(j)} = \frac{i\beta}{a_j^2};$$

$$a_j = \sqrt{k_j^2 - \beta^2}, \quad 0 \le \arg a_j < \pi, \quad k_j = \omega\sqrt{\varepsilon_j'\mu_j}, \quad 0 \le \arg k_j < \pi;$$

and ω is the circular frequency of the field with a time dependence exp $(-i\omega t)$. The functions $w_j = u_j$, v_j satisfy the Helmholtz equation

$$\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} + a_j^2 w_j = 0 \tag{5}$$

in the regions D_j.

Using relations (4), we will transform relations (3) into the boundary conditions in the cylindrical coordinates for the potentials u_j and v_j on the surface Γ ($\rho = R$):

$$\begin{aligned} u_1 - u_2 &= p_1 a^{(1)} \frac{\partial u_1}{\partial \rho} + p_2 a^{(2)} \frac{\partial u_2}{\partial \rho} + p_1 \frac{c^{(1)}}{R}\frac{\partial v_1}{\partial \varphi} + p_2 \frac{c^{(2)}}{R}\frac{\partial v_2}{\partial \varphi}, \\ q_1 u_1 + q_2 u_2 &= a^{(1)} \frac{\partial u_1}{\partial \rho} - a^{(2)} \frac{\partial u_2}{\partial \rho} + \frac{c^{(1)}}{R}\frac{\partial v_1}{\partial \varphi} - \frac{c^{(2)}}{R}\frac{\partial v_2}{\partial \varphi}, \\ v_1 - v_2 &= q_1 b^{(1)} \frac{\partial v_1}{\partial \rho} + q_2 b^{(2)} \frac{\partial v_2}{\partial \rho} - q_1 \frac{c^{(1)}}{R}\frac{\partial u_1}{\partial \varphi} - q_2 \frac{c^{(2)}}{R}\frac{\partial u_2}{\partial \varphi}, \\ p_1 v_1 + p_2 v_2 &= b^{(1)} \frac{\partial v_1}{\partial \rho} - b^{(2)} \frac{\partial v_2}{\partial \rho} - \frac{c^{(1)}}{R}\frac{\partial u_1}{\partial \varphi} + \frac{c^{(2)}}{R}\frac{\partial u_2}{\partial \varphi}. \end{aligned} \tag{6}$$

Scalar boundary problem. It is proposed to determine the functions u_j, $v_j \in C^2(\Omega_j) \cap C^1(\overline{\Omega}_j)$, satisfying Eqs. (5) in the regions Ω_j and the boundary conditions (6); in this case, for the functions $w = u_2$, v_2 the following condition are fulfilled at infinity:

$$\lim_{\rho\to\infty} w(\rho) = 0, \tag{7}$$

where Ω_j are plane regions (the sections of the cylindrical regions D_j formed by the plane $z = 0$). Note that to each propagation constant of the two types $\beta = \beta_{mn}^{(1)}$ and $\beta = \beta_{mn}^{(2)}$ ($m = 0, 1, 2, ...$; $n = 1, 2, ...$) corresponds a solution of problem (5)–(7), called a waveguide mode.

Solution of the Problem. The solution of the waveguide problem is reduced to the solution of an algebraic dispersion equation and to the determination of the propagation constants of a discrete sequence of waveguide electromagnetic modes.

Dispersion equation. For the purpose of solving problem (5)–(7) and calculating the propagation constants β we will construct a dispersion equation. Let us consider the solutions of Eqs. (5), obtained by the method of separation of variables in the polar coordinates, and represent the desired potentials in the form

$$\begin{aligned} &u_1 = U_1 J_m(a_1\rho)\exp(im\varphi), \quad v_1 = U_3 \frac{\omega\varepsilon_1'}{k_1} J_m(a_1\rho)\exp(im\varphi), \quad 0 \le \rho < R; \\ &u_2 = U_2 H_m^{(1)}(a_2\rho)\exp(im\varphi), \quad v_2 = U_4 \frac{\omega\varepsilon_1'}{k_1} H_m^{(1)}(a_2\rho)\exp(im\varphi), \quad \rho > R, \quad m = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{8}$$

Substitution of (8) into the boundary conditions (6) at $\rho = R$ gives a homogeneous system of algebraic equations in U_l:

$$\sum_{l=1}^{4} Q_{kl} U_l = 0\,, \quad k = 1, 2, 3, 4\,. \tag{9}$$

Introducing the designations

$$z = Ra_1\,, \quad \alpha_j = i\omega\varepsilon'_j\,, \quad \delta_j = i\omega\mu_j\,, \quad \tilde{q}_j = \frac{q_j}{R}\,, \quad \tilde{p}_j = Rp_j\,;$$

$$Ra_2 = f(z) \equiv \sqrt{R^2 (k_2^2 - k_1^2) + z^2}\,, \quad 0 \le \arg f(z) < \pi\,;$$

$$R\beta = g(z) \equiv \sqrt{(Rk_1)^2 - z^2}\,, \quad 0 \le \arg g(z) < \pi\,,$$

we obtain expressions for the matrix elements $Q_{kl}(z)$ of the system of equations (9):

$$Q_{11} = \frac{\tilde{p}_1 \alpha_1}{z} J'_m(z) - J_m(z)\,, \quad Q_{12} = \frac{\tilde{p}_2 \alpha_2}{f(z)} H_m^{(1)'}(f(z)) + H_m^{(1)}(f(z))\,,$$

$$Q_{13} = i\tilde{p}_1 \frac{m\alpha_1 g(z)}{k_1 R z^2} J_m(z)\,, \quad Q_{14} = i\tilde{p}_2 \frac{m\alpha_1 g(z)}{k_1 R f^2(z)} H_m^{(1)}(f(z))\,,$$

$$Q_{21} = R\left(\frac{\alpha_1}{z} J'_m(z) - \tilde{q}_1 J_m(z)\right), \quad Q_{22} = -R\left(\frac{\alpha_2}{f(z)} H_m^{(1)'}(f(z)) + \tilde{q}_2 H_m^{(1)}(f(z))\right),$$

$$Q_{23} = i\alpha_1 \frac{mg(z)}{k_1 z^2} J_m(z)\,, \quad Q_{24} = -i\alpha_1 \frac{mg(z)}{k_1 f^2(z)} H_m^{(1)}(f(z))\,,$$

$$Q_{31} = \tilde{q}_1 \frac{mg(z)R}{z^2} J_m(z)\,, \quad Q_{32} = \tilde{q}_2 \frac{mg(z)R}{f^2(z)} H_m^{(1)}(f(z))\,,$$

$$Q_{33} = -\frac{i\alpha_1}{k_1}\left(\frac{\tilde{q}_1 \delta_1 R^2}{z} J'_m(z) - J_m(z)\right), \quad Q_{34} = -\frac{i\alpha_1}{k_1}\left(\frac{\tilde{q}_2 \delta_2 R^2}{f(z)} H_m^{(1)'}(f(z)) + H_m^{(1)}(f(z))\right),$$

$$Q_{41} = \frac{mg(z)}{z^2} J_m(z)\,, \quad Q_{42} = -\frac{mg(z)}{f^2(z)} H_m^{(1)}(f(z))\,,$$

$$Q_{43} = -\frac{i\alpha_1}{k_1 R}\left(\frac{\delta_1 R^2}{z} J'_m(z) - \tilde{p}_1 J_m(z)\right), \quad Q_{44} = \frac{i\alpha_1}{k_1 R}\left(\frac{\delta_2 R^2}{f(z)} H_m^{(1)'}(f(z)) + \tilde{p}_2 H_m^{(1)}(f(z))\right).$$

System (9) has a nontrivial solution U_l (l = 1, 2, 3, 4) if its determinant $D(z)$ is equal to zero. As a result, we obtain the dispersion equation

$$D(z) \equiv \left| Q_{kl}(z) \right| = 0 \tag{10}$$

for determining the two series of roots $z = z_{mn}^{(1)}$ and $z = z_{mn}^{(2)}$ (n = 1, 2, ...). The roots $z_{mn}^{(1)}$ determine the E eigenmodes, $\beta_{mn}^{(1)} = \frac{1}{R} g(z_{mn}^{(1)})$. The complex numbers $z_{mn}^{(1)}$ are close to the roots ν_{mn} of the equation $J_m(\nu_{mn}) = 0$ (n = 1, 2, ...), i.e., $z_{mn}^{(1)} = \nu_{mn} + \varepsilon_{mn}^{(1)}$. For determining these roots, the transcendental equation (10) is solved with the use of the initial approximation $z = \nu_{mn}$. The roots $z_{mn}^{(2)}$ determine the H eigenmodes, $\beta_{mn}^{(2)} = \frac{1}{R} g(z_{mn}^{(2)})$. The complex numbers $z_{mn}^{(2)}$ are close to the roots μ_{mn} of the equation $J_m'(\mu_{mn}) = 0$ (n = 1, 2), i.e., $z_{mn}^{(2)} = \mu_{mn} + \varepsilon_{mn}^{(2)}$.

The roots ν_{mn} and μ_{mn} determine the natural waves of a cylindrical waveguide with an ideally conducting wall [1]. As the first approximation of the solutions of Eq. (10), the eigennumbers of the waveguide with ideally conducting walls were used because the walls of the layered waveguide being investigated have a high conductivity. For an experiment, as the initial approximation, we used the roots of the equation for a single-layer thin-walled waveguide with corresponding electric parameters. The final results were in agreement.

Attenuation coefficients of a field. To calculate the attenuation coefficient of the field of the mode with a double number mn, penetrating through the wall of the layered waveguide, we will compare the amplitudes of all electromagnetic models inside and outside the waveguide. For the construction of the E eigenmodes, system (9) will be solved on the assumption that $z = z_{mn}^{(1)}$. Since system (9) consists of dependent equations (the determinant $D(z_{mn}^{(1)}) = 0$), we drop the first equation and assume that $U_1 = 1$. As a result, we obtain a system for determining the constants U_2, U_3, and U_4:

$$Q_{22}^{(1)} U_2 + Q_{23}^{(1)} U_3 + Q_{24}^{(1)} U_4 = -Q_{21}^{(1)}, \quad Q_{32}^{(1)} U_2 + Q_{33}^{(1)} U_3 + Q_{34}^{(1)} U_4 = -Q_{31}^{(1)},$$

$$Q_{42}^{(1)} U_2 + Q_{43}^{(1)} U_3 + Q_{44}^{(1)} U_4 = -Q_{41}^{(1)},$$

where $Q_{kl}^{(1)} = Q_{kl}(z_{mn}^{(1)})$. The values of $U_1 = 1$, $U_2 = U_{2mn}^{(2)}$, $U_3 = U_{3mn}^{(1)}$, and $U_4 = U_{4mn}^{(1)}$, determined from the above equations, are substituted into formulas (8), from which the z components u_1 and u_2 of the electric field are determined. The attenuation coefficient of the electric component of the E mode is determined as

$$K_{mn}^{\text{el}} = \frac{|u_2(\rho = 2R)|}{\max |u_1|} = \frac{|U_{2mn}^{(1)} H_m^{(1)}(2f(z_{mn}^{(1)}))|}{\max\limits_{0 \le \rho \le R} \left| J_m\left(\frac{\rho}{R} z_{mn}^{(1)}\right)\right|},$$

where $|u_2(\rho = 2R)|$ is the amplitude of the z component of the electric field outside the waveguide at a distance R from the surface Γ and $\max |u_1|$ is the maximum value of the amplitude of the z component of the electric field inside the waveguide.

To construct the H eigenmodes, we will solve system (9) on the assumption that $z = z_{mn}^{(2)}$. The last equation of system (9) is dropped and it is assumed that $U_3 = 1$. As a result, a system of algebraic equations for determining the quantities U_1, U_2, and U_4 is obtained:

$$Q_{11}^{(2)} U_1 + Q_{12}^{(2)} U_2 + Q_{14}^{(2)} U_4 = -Q_{13}^{(2)}, \quad Q_{21}^{(2)} U_1 + Q_{22}^{(2)} U_2 + Q_{24}^{(2)} U_4 = -Q_{23}^{(2)},$$

$$Q_{31}^{(2)} U_1 + Q_{32}^{(2)} U_2 + Q_{34}^{(2)} U_4 = -Q_{33}^{(2)},$$

where $Q_{kl}^{(2)} = Q_{kl}(z_{mn}^{(2)})$. Substituting the values of $U_1 = U_{1mn}^{(2)}$, $U_2 = U_{2mn}^{(2)}$, $U_3 = 1$, and $U_4 = U_{4mn}^{(2)}$, determined from the above formulas, into formulas (8), we find the z components v_1 and v_2 of the magnetic field.

The attenuation coefficient of the magnetic component of the H mode is equal to

$$K_{mn}^{\text{mag}} = \frac{|v_2(\rho = 2R)|}{\max |v_1|} = \frac{|U_{4mn}^{(2)} H_m^{(1)}(2f(z_{mn}^{(2)}))|}{\max\limits_{0 \le \rho \le R} \left| J_m\left(\frac{\rho}{R} z_{mn}^{(2)}\right)\right|}.$$

Simulation of Averaged Boundary Conditions. The algorithm derived above can be used for calculating the coefficients appearing in the averaged boundary conditions and accounting for the electromagnetic properties of the material of all the layers of a waveguide.

Calculation of the coefficients p_j and q_j. The coefficients p_j and q_j, entering into the boundary conditions (3), are calculated in the locally-plane-wave approximation, i.e., the wall of the waveguide is considered at each its point as a locally plane layered structure. It is also assumed that the electromagnetic field inside the multilayer structure propagates along the normal **n** to the surface of the waveguide because the refraction index of the waveguide walls is much larger than the refraction index of the environment. In view of these assumptions, we write the relation between the tangential components of the field at the surface of the s-th layer in terms of the transmission matrix A_s with the use of the following parameters of the layered structure: $\varepsilon'_s = \varepsilon_s^{\rm lay} + i\frac{\gamma_s^{\rm lay}}{\omega}$, $k_s^{\rm lay} = \omega\sqrt{\varepsilon'_s \mu_s^{\rm lay}}$, $0 < \arg k_s^{\rm lay} < \pi$, $Z_s = \frac{\omega\mu_s^{\rm lay}}{k_s^{\rm lay}}$ is impedance of the sth *layer,* $z_r = \sum_{s=1}^{r-1} \Delta_s$, $z_1 = 0$, $r = 2, 3, \ldots, N+1$, $\mathbf{E}_\tau^s = [\mathbf{n}, [\mathbf{E}, \mathbf{n}]]\big|_{z=z_s}$, $\mathbf{H}_\nu^{(s)} = [\mathbf{H}, \mathbf{n}]\big|_{z=z_s}$, $\mathbf{E}_\tau^{(N+1)} = \mathbf{E}_{2\tau}\big|_{\Gamma_2}$, $\mathbf{E}_\tau^{(1)} = \mathbf{E}_{1\tau}\big|_{\Gamma_1}$, then we have

$$\begin{pmatrix} \mathbf{E}_\tau^{(s+1)} \\ \mathbf{H}_\nu^{(s+1)} \end{pmatrix} = A_s \begin{pmatrix} \mathbf{E}_\tau^{(s)} \\ \mathbf{H}_\nu^{(s)} \end{pmatrix}, \quad A_s = \begin{pmatrix} \cos(k_s^{\rm lay}\Delta_s) & iZ_s \sin(k_s^{\rm lay}\Delta_s) \\ \frac{i}{Z_s}\sin(k_s^{\rm lay}\Delta_s) & \cos(k_s^{\rm lay}\Delta_s) \end{pmatrix}.$$

As a result, the relation between the tangential components of the field on each side of the layered structure takes the form

$$\begin{pmatrix} \mathbf{E}_\tau^{(N+1)} \\ \mathbf{H}_\nu^{(N+1)} \end{pmatrix} = B \begin{pmatrix} \mathbf{E}_\tau^{(1)} \\ \mathbf{H}_\nu^{(1)} \end{pmatrix}, \tag{11}$$

where $B = A_N A_{N-1} \ldots A_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is the transmission matrix of the N-layer structure. Relation (11) is reduced, because of the continuity of the tangential components of the field at the medium-medium interfaces, to the relations

$$\mathbf{E}_{2\tau} = b_{11}\mathbf{E}_{1\tau} + b_{12}\mathbf{H}_{1\nu}, \quad \mathbf{H}_{2\nu} = b_{21}\mathbf{E}_{1\tau} + b_{22}\mathbf{H}_{1\nu},$$

equivalent to expressions (3), from where

$$q_1 = b_{21} + (1 - b_{22})\frac{b_{11}}{b_{12}}, \quad q_2 = \frac{b_{22} - 1}{b_{12}}; \quad p_1 = b_{12} + (1 - b_{11})\frac{b_{22}}{b_{21}}, \quad p_2 = \frac{b_{11} - 1}{b_{21}}.$$

Another algorithm for calculating the coefficients p_j and q_j is presented in [8, p. 123].

Averaged Boundary Conditions for Modes. The approximate boundary conditions (3) were used in the calculations for all the modes of the waveguide; therefore, the calculation error increased with increase in the azimuthal number m. The accuracy of calculations can be increased by introduction of individual boundary conditions for each mode:

$$\left[\mathbf{H}_2 - \mathbf{H}_1, \mathbf{n}\right] = \hat{q}_1(\beta, m)\,\mathbf{E}_{1\tau} + \hat{q}_2(\beta, m)\,\mathbf{E}_{2\tau}, \quad \left[\mathbf{E}_2 - \mathbf{E}_1, \mathbf{n}\right] = -\left(\hat{p}_1(\beta, m)\,\mathbf{H}_{1\tau} + \hat{p}_2(\beta, m)\,\mathbf{H}_{2\tau}\right), \tag{12}$$

where $\hat{p}_j$ and $\hat{q}_j$ are matrices of dimension 2×2 written in the basis $\mathbf{e}_\varphi$, $\mathbf{e}_z$ ($\mathbf{n} = \mathbf{e}_\rho$). To calculate the matrices $\hat{p}_j$ and $\hat{q}_j$ in the case of the plane fields $\mathbf{E}, \mathbf{H} = \mathbf{C}\exp(i\lambda_1 x + i\lambda_2 y \pm \nu_s z)$, $\nu_s = \sqrt{\lambda_1^2 + \lambda_2^2 - (k_s^{\rm lay})^2}$, $-\pi/2 \le \arg \nu_s < \pi/2$, we determine the transmission block matrix $\tilde{A}_s$ for the sth layer of thickness Δ_s:

$$\begin{pmatrix} \mathbf{E}_\tau^{(s+1)} \\ \mathbf{H}_\nu^{(s+1)} \end{pmatrix} = \tilde{A}_s \begin{pmatrix} \mathbf{E}_\tau^{(s)} \\ \mathbf{H}_\nu^{(s)} \end{pmatrix}, \quad \tilde{A}_s = \begin{pmatrix} \hat{a}_s & \hat{b}_s \\ \hat{c}_s & \hat{a}_s \end{pmatrix}, \quad s = 1, 2, \ldots, N,$$

TABLE 1. Solution of Dispersion Equations (10) for the Harmonic $m = 0$

n	E mode			H mode		
	I	II	III	I	II	III
1	2.4048	2.3951–0.2391i	2.3951–0.2395i	3.8317	3.8253–0.0251i	3.8254–0.0250i
2	5.5201	5.5022–0.0896i	5.5022–0.0898i	7.0156	7.0042–0.0469i	7.0042–0.0468i
3	8.6537	8.6414–0.0550i	8.6414–0.0551i	10.1735	10.1573–0.0694i	10.1572–0.0693i
4	11.7915	11.7822–0.0397i	11.7822–0.0398i	13.3237	13.3031–0.0930i	13.3028–0.0928i
5	14.9309	14.9235–0.0310i	14.9235–0.0311i	16.4706	16.4459–0.1176i	16.4451–0.1173i

Note. I denotes the initial approximation, II — the classical boundary conditions, III — the averaged boundary conditions.

where

$$\hat{a}_s = \cosh(v_s\Delta_s)\,\hat{e}\,;\quad \hat{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\quad \mathbf{H}_\nu = \hat{s}\mathbf{H}_\tau\,,\quad \hat{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

$$\hat{b}_s = iZ_s \sinh(v_s\Delta_s)\,M_s\,;\quad \hat{c}_s = \frac{i}{Z_s}\sinh(v_s\Delta_s)\,L_s\,;$$

$$M_s = \frac{1}{k_s^{\mathrm{lay}} v_s}\begin{pmatrix} (k_s^{\mathrm{lay}})^2 - \lambda_1^2 & -\lambda_1\lambda_2 \\ -\lambda_1\lambda_2 & (k_s^{\mathrm{lay}})^2 - \lambda_2^2 \end{pmatrix};\quad L_s = \frac{1}{k_s^{\mathrm{lay}} v_s}\begin{pmatrix} (k_s^{\mathrm{lay}})^2 - \lambda_2^2 & \lambda_1\lambda_2 \\ \lambda_1\lambda_2 & (k_s^{\mathrm{lay}})^2 - \lambda_1^2 \end{pmatrix}.$$

As a result, we obtain the relation between the tangential components of the field on each side of the layered structure in the form

$$\begin{pmatrix} \mathbf{E}_{2\tau} \\ \mathbf{H}_{2\nu} \end{pmatrix} = \tilde{B}\begin{pmatrix} \mathbf{E}_{1\tau} \\ \mathbf{H}_{1\nu} \end{pmatrix},\quad \mathbf{E}_{2\tau} = \hat{b}_{11}\mathbf{E}_{1\tau} + \hat{b}_{12}\mathbf{H}_{1\nu}\,,\quad \mathbf{H}_{2\nu} = \hat{b}_{21}\mathbf{E}_{1\tau} + \hat{b}_{22}\mathbf{H}_{1\nu}\,, \tag{13}$$

where $\tilde{B} = \tilde{A}_N\tilde{A}_{N-1}\ldots\tilde{A}_1 = \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{pmatrix}$ is the transmission block matrix of the N-layer structure. Relations (13) are equivalent to conditions (12), from which

$$\hat{q}_1(\beta, m) = \hat{b}_{21} + \left(\hat{e} - \hat{b}_{22}\right)\hat{b}_{22}^{-1}\,\hat{b}_{11}\,,\quad \hat{q}_2(\beta, m) = \left(\hat{b}_{22} - \hat{e}\right)\hat{b}_{12}^{-1}\,;$$

$$\hat{p}_1(\beta, m) = -\hat{s}\left(\hat{b}_{12} + \left(\hat{e} - \hat{b}_{11}\right)\hat{b}_{21}^{-1}\,\hat{b}_{22}\right)\hat{s}\,,\quad \hat{p}_2(\beta, m) = -\hat{s}\left(\hat{b}_{11} - \hat{e}\right)\hat{b}_{21}^{-1}\,,$$

where $\lambda_1 = m/R$ and $\lambda_2 = \beta_{mn}^{(1)}, \beta_{mn}^{(2)}$.

For a single-layer structure ($N = 1$), we have

$$\hat{p}_1 = \hat{p}_2 = iZ_1 \tanh\left(\frac{v_1\Delta_1}{2}\right)L_1\,,\quad \hat{q}_1 = \hat{q}_2 = \frac{i}{Z_1}\tanh\left(\frac{v_1\Delta_1}{2}\right)L_1\,.$$

Note that the boundary conditions (12) account for the slope of the propagation of a field in the layered structure, and, at $\lambda_1 \approx \lambda_2 \approx 0$ and $(|\lambda_1|^2 + |\lambda_2|^2 << |k_s^{\mathrm{lay}}|^2)$, they are transformed into condition (3).

Numerical Results. The roots of the dispersion equation (10) are determined by numerical minimization of the modulus of its left side as a function of the two variables $x = \mathrm{Re}\,z$ and $y = \mathrm{Im}\,z$ with the initial approximation $(v_{mn}, 0)$ for the E modes and the initial approximation $(\mu_{mn}, 0)$ for the H modes. The results obtained for a single-layer waveguide, were compared with the exact solution obtained with the use of the classical boundary conditions. Note that, even for the single-layer waveguide, the exact dispersion equation represents a determinant of the eighth order, the elements of which, unlike (10), involve Bessel functions of complex quantities large in absolute value and, therefore, they are very difficult to calculate.

TABLE 2. Attenuation Coefficients $|K_{mn}^{el}|$ for the E Mode and $|K_{mn}^{mag}|$ for the H Mode

n	m					
	0	1	2	3	4	5
			K_{mn}^{el} *for the E* Mode			
1	0.08959	0.04744	0.04428	0.023334	0.01828	0.01760
2	0.04744	0.02372	0.02180	0.01787	0.02043	0.01422
3	0.02172	0.02931	0.01651	0.01421	0.01148	0.01434
4	0.01647	0.01418	0.01629	0.01483	0.00994	0.00954
5	0.01478	0.01486	0.01054	0.00954	0.01044	0.00800
			K_{mn}^{mag} *for the H* Mode			
1	0.00800	0.03469	0.03042	0.02714	0.02490	0.02329
2	0.01479	0.01195	0.01528	0.01788	0.01998	0.02175
3	0.02150	0.01808	0.02105	0.02357	0.02599	0.02809
4	0.02821	0.02472	0.02766	0.03034	0.03279	0.03507
5	0.03493	0.03143	0.03440	0.03716	0.03972	0.04212

Table 1 presents roots of the exact dispersion equation and of Eq. (10) for the single-layer waveguide with $R = 0.01$ m and $\Delta_1 = 5 \cdot 10^{-5}$ at $\gamma_1^{lay} = 1000$ S/m; the other parameters are identical to those of vacuum, and the linear frequency of the field $f = 4 \cdot 10^{10}$ Hz. It is seen that the roots of the dispersion equations have a negative imaginary part, and the solutions of the exact dispersion equation are in good agreement with the solutions of (10). Our numerical experiments have shown that the differences between the exact and averaged boundary conditions for these parameters are insignificant at $n = 1$–10 m, $m = 0$–10, and $\gamma_1^{lay} > 100$ S/m.

Below are results of the calculations carried out for a four-layer waveguide with an averaged radius $R = 0.01$ m and a wall of thickness $\Delta_i = 5 \cdot 10^{-5}$, $i = 1, 2, 3, 4$. The layers of the waveguide have the following parameters: $\varepsilon_1 = \varepsilon_3 = 10\varepsilon_0$, $\varepsilon_2 = \varepsilon_4 = 100\varepsilon_0$, $\mu_s = \mu_0$, $s = 1, 2, 3, 4$; $\gamma_1^{lay} = \gamma_3^{lay} = 10$ S/m, $\gamma_2^{lay} = \gamma_4^{lay} = 1000$ S/m.

It is assumed that the medium inside and outside the waveguide is a vacuum, and the linear frequency of the field $f = 4 \cdot 10^{10}$ Hz. In Table 2, the moduli of the attenuation coefficients of the E-mode electric component and the H-mode magnetic component are presented [10]. Note that, by and large, when n and m increase, the attenuation coefficient of the E mode decreases and the attenuation coefficient of the H mode increases.

CONCLUSIONS

1. The results of calculations of the attenuation coefficients and the propagation constants with the use of the classical boundary conditions and the averaged boundary conditions agree well for a single-layer waveguide at $n = 1$–10 and $m = 0$–10.

2. The calculation of the spectral parameters for a multilayer waveguide with the use of exact boundary conditions calls for the calculation of determinants of large orders and Bessel functions of complex arguments large in absolute value, which can be obviated in the case where the averaged boundary conditions are used.

3. The results of calculations of the coefficients entering into the averaged boundary conditions by the two different algorithms used in the present investigation are in complete agreement.

4. The method developed by us can be used for designing waveguide communication lines with optimization of the electromagnetic field emitting by them into the environment.

NOTATION

A_s, B, M_s, L_s, $\hat{p}_j$, $\hat{q}_j$, $\hat{a}_s$, $\hat{b}_s$, $\hat{c}_s$, $\hat{b}_{11}$, $\hat{b}_{12}$, $\hat{b}_{21}$, $\hat{b}_{22}$, matrices of dimension 2×2; $\tilde{A}_s$, $\tilde{B}_s$, $\tilde{B}$, matrices of dimension 4×4; $D(z)$, determinant; $\mathbf{E}^{(s)}$, $\mathbf{H}^{(s)}$, electromagnetic field in the layer s; $\mathbf{E}_1$, $\mathbf{E}_2$, complex amplitudes of the strength of the electric fields inside the waveguide and outside it, V/m; $f(z)$, $g(z)$, a_j, p_j, q_j, $\tilde{p}_j$, $\tilde{q}_j$, w, w_j, α_j, δ_j, auxiliary functions and constants; $\mathbf{H}_1$, $\mathbf{H}_2$, complex amplitudes of the strength of the magnetic fields inside the waveguide and outside it, A/m; $i = \sqrt{-1}$, complex unity; $J_m(\cdot)$, Bessel function of the first kind; $J_m'(\cdot)$, derivative of the Bessel function; K_{mn}^{el}, K_{mn}^{mag}, attenuation coefficients of the E- and H-polarized modes of the waveguide with a double number mn; k_s^{lay}, wave number of the sth layer of the waveguide wall, 1/m; k_j, wave number inside and outside the waveguide, 1/m; Q_{kl}, elements of the matrix of dimension 4 × 4; R, average radius of the waveguide wall, m; R_1, R_2, inner and outer radii of the waveguide wall, m; u_j, v_j, zth projections of the strength vectors of the electric and magnetic fields, A/m, V/m; Z_s, impedance of the sth layer of the waveguide wall; $z = z_{mn}^{(1)}$, $z_{mn}^{(2)}$, complex roots of the dispersion equation; $\beta = \beta_{mn}^{(1)}$, $\beta_{mn}^{(2)}$, propagation constants of the waveguide, 1/m; γ_j, electric conductivity of the media inside and outside the waveguide, S/m; γ_s^{lay}, electric conductivity of the sth layer of the waveguide wall; Δ_s, thickness of the sth layer of the waveguide wall, m; ε_j, permittivity of the media inside and outside the waveguide, F/m; $\varepsilon_s^{\text{lay}}$, permittivity of the sth layer of the waveguide wall; ε_j', complex permittivity; μ_j, magnetic permeability of the media inside and outside the waveguide, H/m; μ_s^{lay}, magnetic permeability of the sth layer of the waveguide wall; ρ, φ, z, cylindrical coordinates; $\omega = 2\pi f$, f, frequency of a field, 1/sec. Subscripts: j = 1, 2: 1, medium inside the waveguide; 2, medium outside the waveguide; s, number of a layer (s = 1, 2, ..., N); lay, layer; mn, double number of a mode; el, electrical; mag, magnetic; τ, ν, tangential components of a field.

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